B.A./B. Sc. (Part III) Examination, 2012

MATHEMATICS : Second Paper

(Abstract Algebra)

Note : Attempt questions from all sections.

Section - A

Note : Attempt any six questions. Each question carries equal marks. (20/30)

1. Define centre of a group. Prove that the centre Z of a group G is a normal subgroup of G.
2. If $O(G) = p^2$ where p is a prime number, show that the centre $Z \neq \{e\}$.
3. Prove that the intersection of two ideals of a ring R is again on ideal of R.
4. If a commutative ring with unity has no proper ideal then prove that it is a field.
5. Define Euclidean field is a Euclidean ring.
6. Prove that every field is a Euclidean ring.
7. Define basis of a vector space. Show that the set $S = \{1, x, x^2, ... x^n\}$ of $n + 1$ polynomials in x is a basis of the vector space $P_n(R)$ of all polynomials in x over the field of real number.
8. If W_1, W_2 are two subspaces of a finite dimensional vector space V then prove that :
   $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
9. Define annihilator. Show that annihilator of a subset of vector space V is a subspace of dual space V′.
10. Show that the mapping $T : V_2(R) \to V_3(R)$ defined by
    $T(a, b) = (a + b, a - b, b)$ is a linear transformation from $V_2(R)$ into $V_3(R)$. Find null space of T also.

Section - B

Note : Attempt any two questions. Each questions carries equal marks. (30/45)

1. (a) Show that the mapping $a \to a^{-1}$ is an automorphism of a group G iff G is abelian.

   (b) Prove that the group of automorphism of an infinite cycic group is of order 2.
2. (a) Prove that the polynomial ring $D[x]$ of an integral domain is again an integral domain.

   (b) Let $f(x) = 2x^4 + 3x^3 + 2$ and $g(x) = 4x + 3$ be two polynomials over the field $Z_5 = (\{0, 1, 2, 3, 4\}, x_5, x_5)$. Find $f(x) . g(x)$.
3. If a finite dimensional vector space V is a direct sum of its two subspaces W_1 and W_2 then prove that
   $\dim V = \dim W_1 + \dim W_2$
   Also prove that if $V = W_1 + W_2$ then $V = W_1 \oplus W_2$

4. Let T be a linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

Prove that T is invertible and final a formula for T^{-1}.

Or

Let T be a linear transformation from a finite dimensional space U (F) into a vector space V (F). Then prove that rank (T) + nullity (T) = dim U.